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Holden in 1901 showed that in about half of the engineering schools of the United States, attention is given to the use of the slide rule.

A BIQUADRATIC EQUATION CONNECTED WITH THE REDUCTION OF A QUADRATIC LOCUS.

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If the equation of a conic section be written in the form

$$Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey + F = 0,$$

then it is known that a rotation of the coordinate axes through an angle α will bring them into parallelism with the axes of symmetry of the curve, provided this angle is determined by

$$\tan 2\alpha = \frac{2C}{A-B}.$$

This rotation corresponds to the substitution

$$(1) \quad \begin{aligned} x &= x' \cos \alpha - y' \sin \alpha, \\ y &= y' \cos \alpha + x' \sin \alpha, \end{aligned}$$

with α so chosen as to eliminate the term in $x'y'$. But the sine and cosine may be expressed in terms of the tangent of the half-angle, thus:

$$(2) \quad t = \tan \frac{\alpha}{2}, \quad \cos \alpha = \frac{1-t^2}{1+t^2}, \quad \sin \alpha = \frac{2t}{1+t^2},$$

and the use of these in (1) gives the substitution expressed rationally in terms of the parameter t . Without reference to its trigonometric source, the substitution in that form is seen to be orthogonal or rotational for all values of t , since the equation of constancy of distances:

$$(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

is directly verifiable as an identity in t .

The use of this parameter makes it possible to effect the reduction of the conic by purely algebraic processes, independently of the trigonometric formulae. For the term in $x'y'$ will have as coefficient

$$\frac{2}{(1+t^2)^2} \{2(B-A) \cdot t(1-t^2) + C[(1-t^2)^2 - (2t)^2]\},$$

which will vanish if t satisfy the biquadratic equation:

$$(3) \quad t^4 + 4mt^3 - 6t^2 - 4mt + 1 = 0,$$

in which is put $m = \frac{A-B}{2C}$, the trigonometric value of which is $\cot 2\alpha$. The four roots of this equation must therefore be real for all real values of m , and correspond to the four semi-axes of the conic.

But this equation must be solvable by quadratics. For, by a familiar formula of trigonometry

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

which, regarded as a quadratic equation in $\tan \alpha$ gives

$$\tan \alpha = -\cot 2\alpha \pm \sqrt{1 + \cot^2 2\alpha},$$

and a repetition of such solution gives $\tan \frac{1}{2}\alpha$ in terms of $\cot \alpha$. This suggests at once the four roots of the biquadratic in t , which after a little reduction prove to be:

$$\begin{array}{ll} t_1 = -m + r + R_1 & \text{where} \\ t_2 = -m + r + R_2 & r = \sqrt{1 + m^2}, \\ t_3 = -m + r - R_1 & R_1 = \sqrt{2(r^2 - rm)}, \\ t_4 = -m - r - R_2 & R_2 = \sqrt{2(r^2 + rm)}. \end{array}$$

These roots are obviously all real, and are easily shown to be distinct. Direct computation shows that the product $(t - t_1)(t - t_2)(t - t_3)(t - t_4)$ gives the biquadratic polynomial on the left of the equation in t .

The reducibility of the biquadratic equation by quadratics is connected intimately with the existence of rational relations among the roots, which in the present case are the following:

$$t_1 t_3 = -1, \quad t_2 t_4 = -1, \quad \frac{t_2 - t_1}{1 + t_1 t_2} = \frac{t_3 - t_4}{1 + t_3 t_2},$$

and others (not independent) similar to the last. These correspond to the fact that, since the axes of the conic are mutually perpendicular, the various values of $\frac{1}{2}\alpha$ must be spaced at intervals of 45° .